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Another Look at Scattering Theory

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We present a new approach to existence and completeness of wave operators. We do not use the subspace of absolute continuity, but rather the orthogonal complement of the eigenvectors. This is more natural from the physical point of view. We give sufficient conditions for existence and completeness of the wave operators. These results are both simpler and stronger than those obtained previously.

1. INTRODUCTION

In the study of scattering theory one is interested in the limits

$$W_{\pm}\Psi = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}\Psi \tag{1.1}$$

where H, H_0 are self-adjoint operators in a Hilbert space $\mathscr{H}$ and Ψ is an element of $\mathscr{H}$. It is a simple matter to show that it is unreasonable to expect the limits (1.1) to exist when Ψ is an eigenvector of H_0. For then the limits will exist only if Ψ is also an eigenvector of H corresponding to the same eigenvalue. Moreover, mathematicians discovered quite early that it is easier to deal with the limits (1.1) if one takes Ψ to be in the subspace of absolute continuity of H_0 (cf. Kato, 1966a). In particular, one can obtain theorems that need not be true otherwise. However, the subspace of absolute continuity has no real physical significance and seems artificial from the point of view of applications. It is true that, in all cases that have been thoroughly analyzed, it results that the subspace of absolute continuity coincides with the orthogonal complement of the eigenvectors. However, an abstract theory that is based on this premise must verify it in each application or have a gap to fill. It is very likely that a situation will arise where the two subspaces do not coincide.

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The purpose of the present paper is to develop an abstract scattering theory that refers to the subspace of continuity (i.e., the orthogonal complement of the eigenvectors) rather than the subspace of absolute continuity. We obtain criteria for the existence and completeness of the wave operators (1.1). Our theorems are not much more difficult to state (and prove) than the corresponding weaker theorems for the subspace of absolute continuity. Our work generalizes that of Agmon (1975), Kuroda (1973), and Schechter (1976).

In Section 2 we state our main theorems for the case when H is a perturbation of H_0 in a generalized sense and the perturbation can be factored. The proofs are given in Section 3, where we employ a new general theory. For discussions related to this topic we refer to Wilcox (1972), Amrein and Georgescu (1973), and Prugovecki (1971).

2. THE THEORY

We denote the sets of those ψ for which the limits (1.1) exist by $D(W_\pm)$. It is easily checked that they are closed subspaces of $\mathscr{H}$. We take these to be the domains of the wave operators $W_\pm$ defined by (1.1). Let $E(\lambda)$, $E_0(\lambda)$ be the spectral families of H, H_0, respectively. We define the subspace of continuity $\mathscr{H}_c(H)$ of H as the set of those $f \in \mathscr{H}$ such that $(E(\lambda)f, f)$ is a continuous function of λ. It is a closed subspace and coincides with the orthogonal complement of the subspace spanned by all the eigenvectors of H (cf. Kato, 1966a). We shall call the wave operators $W_\pm$ *complete* if $\mathscr{H}_c(H_0) \subset D(W_\pm)$ and their ranges $R(W_\pm)$ coincide. We shall call them *strongly complete* if, in addition, $\mathscr{H}_c(H) \subset R(W_\pm)$. The resolvents of H, H_0 will be denoted by $R(z)$, $R_0(z)$, respectively.

We consider the case when we can find a Hilbert space $\mathscr{K}$ and linear operators A, B from $\mathscr{H}$ to $\mathscr{K}$ such that $D(H_0) \subset D(A)$, $D(H) \subset D(B)$ and

$$(u, Hv) = (H_0u, v) + (Au, Bv)_{\mathscr{K}}, \qquad u \in D(H_0), v \in D(H) \tag{2.1}$$

If Γ is a subset of the real line R and I is an interval, we shall write $I \subset\subset \Gamma$ when I is bounded and $\bar{I} \subset \Gamma$. Our first result is the following theorem.

Theorem 2.1. Assume that (2.1) holds and there is an open set $\Gamma \subset R$ such that $C\Gamma = R - \Gamma$ is denumerable and

$$a\|AR_0(s \pm ia)\|^2 + a\|BR(s \pm ia)\|^2 \leqslant C_I, \qquad s \in I, a > 0 \tag{2.2}$$

holds for all $I \subset\subset \Gamma$. Then the wave operators are strongly complete.

The advantage of Theorem 2.1 is that it is symmetric in H and H_0. However, it has the disadvantage of having its hypotheses involve $R(z)$. In most applications H is a perturbation of H_0 in some sense and it is usually very difficult to compute $R(z)$ or obtain estimates for it. Using methods motivated by ideas from Kato and Kuroda (1971), Agmon (1975), Kuroda (1973) and Schechter (1976) we formulate a version which is easier to apply in such situations. We shall denote the closure of an operator L by $[L]$. The use of this symbol will imply the assumption that the operator is closable.

Theorem 2.2. Suppose A, B are closed and there is a number θ such that $0 \leqslant \theta \leqslant 1$,

$$D(|H_0|^{\theta}) = D(|H|^{\theta}) = D_\theta \subset D(A) \tag{2.3}$$

$$D(|H_0|^{1-\theta}) = D(|H|^{1-\theta}) = D_{1-\theta} \subset D(B) \tag{2.4}$$

and (2.1) holds. For $\operatorname{Im} z \neq 0$ put

$$Q_0(z) = A[BR_0(\bar{z})]^*, \qquad G_0(z) = 1 - Q_0(z) \tag{2.5}$$

Assume that there is a $z_1 \in \rho(H_0)$ such that $AR_0(z)[BR_0(\bar{z}_1)]^*$ is a compact operator on $\mathscr{H}$ for all nonreal z. Assume further that there is an open set Γ such that $C\Gamma$ is denumerable and $G_0(s \pm ia) \to G_{0\pm}(s)$ is norm for each $s \in \Gamma$, where the $G_{0\pm}(s)$ are continuous in s. Also

$$a\|AR_0(s \pm ia)\|^2 + a\|BR_0(s \pm ia)\|^2 \leqslant C_I, \qquad s \in I, a > 0 \tag{2.6}$$

for each $I \subset\subset \Gamma$. Assume further that $D(B^*)$ is dense and for each $g \in N[G_{0\pm}(s)]$ there is a function $\sigma(\delta) \to 0$ as $\delta \to 0$ such that

$$\|[BE_0(I)R_0(s + it)]^*g\| \leqslant \sigma(|I|) \tag{2.7}$$

holds when s is the midpoint of I. Finally assume that there is a locally bounded function $C(s)$ in Γ and functions $\tau_j(\delta) \to 0$ as $\delta \to 0$ such that

$$\begin{aligned}&\|[BE_0(I)R_0(s + it)]^*Au\| \\ &\quad \leqslant C(s)[\tau_1(|I|) + \tau_2(|s - \lambda|/|I|)\|u\|], \qquad u \in N(H - s)\end{aligned} \tag{2.8}$$

where λ is the center of the interval $I \subset\subset \Gamma$. Then the wave operators are strongly complete.

The hypotheses of Theorem 2.2 have been verified for the Schrödinger operator with singular potentials (cf. Agmon, 1975; Kuroda, 1973; Schechter, 1976).

3. THE PROOFS

In proving Theorems 2.1 and 2.2 our starting point will be the following theorem, proved in Schechter (1977a, b). We put

$$j(z, f, g) = \operatorname{Im} z(R_0(z)f, [R(z) - R_0(z)]g)/\pi \tag{3.1}$$

$$J_I(f, g) = \lim_{0<a\to 0} \int_I j(s + ia, f, g)\, ds \tag{3.2}$$

when the limit exists. We have the following

Theorem 3.1. Suppose $f \in \mathscr{H}_c(H_0)$ and that for each bounded interval I there is a dense set $S_I \subset \mathscr{H}$ such that $J_I(f, g)$ exists for each $g \in S_I$. Assume also that

$$J_I(f_t, f_t) \to 0 \qquad \text{as } t \to \infty \tag{3.3}$$

for each bounded I, where $f_t = e^{-itH_0}f$. Then $f \in D(W_+)$.

An immediate consequence of this is

Theorem 3.2. Suppose $f \in \mathscr{H}_c(H_0)$ and there is an open interval Λ such that

(a) For each interval $I \subset\subset \Lambda$ there is a dense subset $S_I \subset \mathscr{H}$ such that $J_I(f, g)$ exists for all $g \in S_I$ and

(b) $$J_I(E_0(I)f_t, E_0(I)f_t) \to 0 \qquad \text{as } t \to \infty \tag{3.4}$$

Then $E_0(\Lambda)f \in D(W_+)$.

Proof. If $I \subset\subset \Lambda$, it is easily seen that

$$J_I(E_0(I)f, g) = J_I(f, g) \tag{3.5}$$

and that

$$J_K(E_0(I)f, g) = 0, \qquad K \cap I = \varnothing \tag{3.6}$$

(cf. Schechter, 1978b). Thus $J_L(E_0(I)f, g)$ exists for every interval L. In particular, (3.6) implies

$$J_K(E_0(I)f_t, E_0(I)f_t) = 0, \qquad K \cap I = \varnothing$$

Thus

$$J_L(E_0(I)f_t, E_0(I)f_t) \to 0 \qquad \text{as } t \to \infty \tag{3.7}$$

for any bounded interval L. Thus $E_0(I)f \in D(W_+)$ by Theorem 3.1. By taking a sequence $I_n \subset\subset \Lambda$ such that $I_n \to \Lambda$, we see that $E_0(\Lambda)f$ is the limit of elements of $D(W_+)$. Since $D(W_+)$ is closed, the result follows. ∎

Theorem 3.3. Suppose (2.1) holds and $f \in \mathcal{H}_c(H)$. Assume that there is an open set Λ such that for each $I \subset\subset \Lambda$

$$\int_0^\infty \|AE_0(I)f_t\|^2\, dt < \infty \tag{3.8}$$

and

$$a\|BR(s - ia)\|^2 \leq C_I, \qquad s \in I,\ a > 0 \tag{3.9}$$

Then $E_0(\Lambda)f \in D(W_+)$.

Proof. It is easily shown that (3.8) implies

$$a\|BR(s - ia)E(\bar{I})\|^2 \leqslant 4C_I \tag{3.10}$$

(cf. Lavine, 1972). This in turn implies

$$\int_{-\infty}^\infty \|Be^{-itH}E(\bar{I})g\|^2\, dt \leqslant 8C_I\|g\|^2$$

(cf. Kato, 1966b). Thus there are functions $f(s), g_\pm(s)$ in $L^2(-\infty, \infty, \mathcal{K})$ such that

$$\int_{-\infty}^\infty \|AR_0(z)E_0(I)f - f(s)\|^2\, ds \to 0 \qquad \text{as } a \to 0 \tag{3.11}$$

and

$$\int_{-\infty}^\infty \|BR(s \pm ia)E(\bar{I})g - g_\pm(s)\|^2\, ds \to 0 \tag{3.12}$$

Moreover, (2.1) implies

$$R(z) = R_0(z) + [BR(\bar{z})]^* AR_0(z) \tag{3.13}$$

Thus

$$\begin{aligned} a\int_I ([R(z) - R_0(z)]E_0(I)f, R(z)E(\bar{I})g)\, ds \\ &= a\int_I (AR_0(z)E_0(I)f, BR(\bar{z})R(z)E(\bar{I})g)\, ds \\ &= -\tfrac{1}{2}i\int_I (AR_0(z)E_0(I)f, B[R(z) - R(\bar{z})]E(\bar{I})g)\, ds \\ &\to -\tfrac{1}{2}i\int_I (f(s), [g_+(s) - g_-(s)])\, ds \qquad \text{as } a \to 0 \end{aligned}$$

On the other hand

$$a\int_I |([R(z) - R_0(z)]E_0(I)f, R(z)E(C\bar{I})g)|\, ds$$

$$\leqslant \left\{a\int_I \|[R(z) - R_0(z)]E_0(I)f\|^2\, ds\right\}^{1/2}$$

$$\cdot\left[a\int_I \|R(z)E(C\bar{I})g\|^2\, ds\right]^{1/2} \to 0 \qquad \text{as } a \to 0$$

Thus $J_I(E_0(I)f, g)$ exists for each $g \in \mathscr{H}$. Also

$$a\int_I |(R_0(z)E_0(I)f_t, [R(z) - R_0(z)]E_0(I)f_t)|\, ds$$

$$= a\int_I |(BR(\bar{z})R_0(z)E_0(I)f_t, AR_0(z)E_0(I)f_t)|\, ds$$

$$\leqslant \left[a^2\int_I \|BR(\bar{z})R_0(z)E_0(I)f_t\|^2\, ds\right]^{1/2}$$

$$\cdot\left[\int_I \|AR_0(z)E_0(I)f_t\|^2\, ds\right]^{1/2}$$

$$\leqslant \left[aC_I\int_I \|R_0(z)E_0(I)f_t\|^2\, ds\right]^{1/2}\left[\int_0^\infty \|Ae^{-i\sigma H_0}E_0(I)f_t\|^2\, d\sigma\right]^{1/2}$$

$$\leqslant \pi^{1/2}C_I^{1/2}\|E_0(I)f\|\left[\int_t^\infty \|AE_0(I)f_\tau\|^2\, d\tau\right]^{1/2}$$

This implies (3.4). Thus by Theorem 3.2, $E_0(\Lambda_0)f \in D(W_+)$ for each component Λ_0 of Λ. This implies that $E_0(\Lambda)f \in D(W_+)$. ∎

Now we can give the following.

Proof of Theorem 2.1. Suppose $f \in \mathscr{H}_c(H_0)$. Inequality (2.2) implies

$$a\|AR_0(z)E_0(I)\|^2 + a\|BR(z)E(I)\|^2 \leqslant 8C_I, \qquad a = |\text{Im } z| \neq 0 \tag{3.14}$$

for $I \subset\subset \Gamma$. This implies

$$\int_{-\infty}^{\infty} \|AE_0(I)f_t\|^2\, dt < \infty \tag{3.15}$$

(cf. Kato, 1966b). All of the hypotheses of Theorem 3.3 are satisfied. Thus $E_0(\Gamma)f \in D(W_+)$. Now by hypothesis

$$C\Gamma = \bigcup_{k=1}^{\infty} \{\lambda_k\}$$

Thus

$$E_0(C\Gamma)f = \sum_{k=1}^{\infty} [E(\lambda_k) - E(\lambda_k -)]f = 0$$

since $f \in \mathscr{H}_c(H_0)$. Thus $E_0(\Gamma)f = f$ and we can conclude that $f \in D(W_+)$. If we consider the pair $-H$, $-H_0$ in place of H, H_0, the hypotheses of Theorem 3.3 are satisfied as well. Thus $f \in D(W_-)$. Finally we note that the hypotheses of Theorem 2.1 are symmetric in H and H_0. Thus the limits

$$\lim_{t \to \pm\infty} e^{itH_0} e^{-itH} g$$

exists for each $g \in \mathscr{H}_c(H)$. This implies that $\mathscr{H}_c(H) \subset R(W_\pm)$. ∎

In proving Theorem 2.2 we shall use the following lemmas.

Lemma 3.4. If $g \in N[G_{0\pm}(\lambda)]$, then there is a $w \in D(H)$ such that $g = Aw$ and $(H - \lambda)w = 0$.

Proof. Let $\epsilon > 0$ be given, and let I be an interval $\subset\subset \Gamma$ with center λ such that $\sigma(|I|) < \epsilon$. Then by (2.7)

$$\|\{B[R_0(\lambda + is) - R_0(\lambda + it)]\}^* g\| < 2\epsilon + \|\{BE_0(CI)[R_0(\lambda + is) - R_0(\lambda + it)]\}^* g\|$$

Thus $[BR_0(\lambda + is)]^* g$ converges to some element w in $\mathscr{H}$ as $s \to 0$. Moreover

$$A[BR_0(\lambda \pm ia)]^* g \to g - G_{0\pm}(\lambda) g = g$$

Since A is closed, we see that $w \in D(A)$ and $Aw = g$. Now

$$([\lambda \pm ia - H_0]u, [BR_0(\lambda \pm ia)]^* g) = (Bu, Aw)$$

for all $u \in D(H_0)$. Letting $a \to 0$ we get

$$([\lambda - H_0]u, w) = (Bu, Aw), \qquad u \in D(H_0) \tag{3.16}$$

In particular, we have

$$|([i - H_0]u, w)| \leqslant C\| |i - H_0|^{1-\theta} u\|$$

This shows that $w \in D_\theta$. From this we see that (3.16) holds for all $u \in D_{1-\theta}$. In fact, for each such u we can put $u_k = E_0(-k, k)u$. Then $u_k \in D(H_0)$ and $u_k \in D(H_0)$ and $u_k \to u$ in $D_{1-\theta}$. Apply (3.16) to u_k and let $k \to \infty$. Now we can apply (2.1) to conclude

$$([\lambda - H]u, w) = 0, \qquad u \in D_{1-\theta}$$

Since $D(H) \subset D_{1-\theta}$, we see that $w \in E(H)$ and $(\lambda - H)w = 0$. ∎

Lemma 3.5. The set of points λ for which $N[G_{0\pm}(\lambda)] \neq \{0\}$ has no limit points in Γ.

Proof. Suppose $g_k \in N[G_{0\pm}(\lambda_k)]$, $\quad g_k \neq 0$
$\lambda_k \to \lambda$, λ_k, $\lambda \in \Gamma$. Then by Lemma 3.4 the λ_k are eigenvalues of H with eigenvectors w_k satisfying $Aw_k = g_k$ obtained as in the Proof of Lemma 3.4. We may assume that $\|w_k\| = 1$. Since

$$g_k = (i + |\lambda_k|^\theta)A(i + |H|^\theta)^{-1}w_k$$

we see that the g_k are uniformly bounded in norm. Hence there is a subsequence (also denoted by $\{g_k\}$) which converges weakly. Put

$$K(z) = (z - z_1)G_0(z_1)^{-1}AR_0(z)[BR_0(\bar{z}_1)]^*$$

where z_1 is the point mentioned in the hypotheses. {Note that $G_0(z_1)^{-1} = 1 + A[BR(\bar{z}_1)]^*$.} Then

$$G_0(z) = G_0(z_1)[1 + K(z)] \tag{3.17}$$

Thus $K(s \pm ia) \to K_\pm(s)$ as $a \to 0$, where the limit functions are compact and depend continuously on s in Γ. By (3.17)

$$g_k = [K_\pm(\lambda) - K_\pm(\lambda_k)]g_k - K_\pm(\lambda)g_k$$

Since g_k converges weakly and the $K_\pm(s)$ are compact and continuous, we see that the g_k converge strongly. Let $\epsilon > 0$ be given, and let I be an interval with center λ containing the λ_k in its interior such that $C(\lambda_k)\tau_1(|I|) < \epsilon$ for all k. Then take the λ_k so close to λ that $C(\lambda_k)\tau_2(|\lambda - \lambda_k|/|I|) < \epsilon$ for all k. Then we have

$$\begin{aligned}\|w_j - w_k\| \leqslant \|w_j &- [BR_0(\lambda_j \pm ia)]^*g_j\| \\ &+ \|[BR_0(\lambda_j \pm ia)]^*g_j - [BR_0(\lambda_k \pm ia)]^*g_k\| \\ &+ \|[BR_0(\lambda_k \pm ia)]^*g_k - w_k\|\end{aligned}$$

The next to the last term is bounded by

$$\begin{aligned}&\|[BE_0(I)R_0(\lambda_j \pm ia)]^*g_j\| + \|[BE_0(I)R_0(\lambda_k \pm ia)]^*g_k\| \\ &\qquad + \|[BE_0(CI)R_0(\lambda_j \pm ia)]^*g_j - [BE_0(CI)R_0(\lambda_k \pm ia)]^*g_k\|\end{aligned}$$

Letting $a \to 0$, we get

$$\|w_j - w_k\| \leqslant 2\epsilon + \|[BE_0(CI)R_0(\lambda_j)]^*g_j - [BE_0(CI)R_0(\lambda_k)]^*g_k\|$$

This shows that the $\{w_k\}$ form a Cauchy sequence. But they are orthonomal, being eigenvectors of a self-adjoint operator corresponding to different eigenvalues. Thus the λ_k cannot converge to a limit in Γ. ∎

Proof of Theorem 2.2. Let e be the set of those $\lambda \in \Gamma$ such that $N[G_{0\pm}(\lambda)] \neq \{0\}$. By Lemma 3.5, e has no limit points in Γ. Thus $\tilde{\Gamma} = \Gamma - e$ is open and $C\tilde{\Gamma}$ is a denumerable set. Next we note that

$$G_0(z)BR(z) = BR_0(z)$$

by (3.13). Moreover, by (3.17)

$$G_{0\pm}(\lambda) = G_0(z_1)[1 + K_\pm(\lambda)]$$

Since the $K_\pm(\lambda)$ are compact, we see that $G_{0\pm}(\lambda)$ has a bounded inverse for $\lambda \in \tilde{\Gamma}$. Hence (2.6) implies (2.2) for $I \subset\subset \Gamma$. We can now apply Theorem 2.1 to obtain the desired conclusion. ∎

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